

Some intrinsic properties of h-Randers conformal change

¹H. S. Shukla, ²V. K. Chaubey and ³Arunima Mishra

¹Department of Mathematics and Statistics, D. D. U. Gorakhpur University, Gorakhpur (U.P.)-273009, India,
E-mail:- profhsshuklagkp@rediffmail.com

²Department of Applied Sciences, Buddha Institute of Technology, Sector-7, Gida, Gorakhpur, (U.P.) India'
E-Mail: vkchaubey@outlook.com

³Department of Mathematics, St. Joseph's College for Women, Civil Lines, Gorakhpur, (U.P.), India,
E-Mail: arunima16oct@hotmail.com

Abstract

In the present paper we have considered h-Randers conformal change of a Finsler metric L , which is defined as

$$L(x, y) \rightarrow \bar{L}(x, y) = e^{\sigma(x)} L(x, y) + \beta(x, y),$$

where $\sigma(x)$ is a function of x , $\beta(x, y) = b_i(x, y)y^i$ is a 1- form on M^n and b_i satisfies the condition of being an h-vector. We have obtained the expressions for geodesic spray coefficients under this change. Further we have studied some special Finsler spaces namely quasi-C-reducible, C-reducible, S3-like and S4-like Finsler spaces arising from this metric. We have also obtained the condition under which this change of metric leads a Berwald (or a Landsberg) space into a space of the same kind.

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1 Introduction

Let M^n be an n -dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function $L(x, y)$, ($y^i = \dot{x}^i$) of M^n . If a differential 1-form $\beta(x, y) = b_i(x)y^i$ is given on M^n , M. Matsumoto [7] introduced another Finsler space whose fundamental function is given by

$$\bar{L}(x, y) = L(x, y) + \beta(x, y)$$

This change of Finsler metric has been called β -change [10, 11].

The conformal theory of Finsler spaces was initiated by M.S. Knebelman [6] in 1929 and has been investigated in detail by many authors [1, 2, 3, 4]. The conformal change is defined as

$$\bar{L}(x, y) \rightarrow e^{\sigma(x)} L(x, y),$$

where $\sigma(x)$ is a function of position only and known as conformal factor.

In 1980, Izumi [3] introduced the h -vector b_i which is v -covariantly constant with respect to Cartan's connection CT (i.e. $b_i|_j = 0$) and satisfies the relation $LC_{ij}^h b_h = \rho h_{ij}$, where C_{ij}^h are components of (h)hv-torsion tensor and h_{ij} are components of angular metric tensor. Thus the h -vector is not only a function of coordinates x^i , but it is also a function of directional arguments satisfying $L\dot{\partial}_j b_i = \rho h_{ij}$.

In the paper [14] S. H. Abed generalized the above two changes and have introduced another Finsler metric named as Conformal β -change and further Gupta and Pandey [15] renamed it Randers conformal change and obtained various important result in the field of Finsler spaces. Recently we [13] have generalized the metric given by S. H. Abed with the help of h -vector and have introduced another Finsler metric which is defined as

$$\bar{L}(x, y) = e^{\sigma(x)} L(x, y) + \beta(x, y), \quad (1.1)$$

where $\sigma(x)$ is a function of x and $\beta(x, y) = b_i(x)y^i$ is a 1-form on M^n and b_i satisfies the condition of being an h -vector, We call the change $L(x, y) \rightarrow \bar{L}(x, y)$ as h -Randers conformal change. This change generalizes various types of changes. When $\beta = 0$, it reduces to a conformal change. When $\sigma = 0$, it reduces to a h -Randers change [9]. When $\beta = 0$ and σ is a non-zero constant then it reduces to a homothetic change. When b_i is function of position only and $\sigma = 0$, it reduces to Randers change [12]. When b_i and σ are functions of position only, it reduces to Randers conformal change

[14, 15].

In the present paper we have obtained the expressions for geodesic spray coefficients under this change. Further we have studied some special Finsler spaces namely quasi C-reducible, C-reducible, S3-like and S4-like Finsler spaces arising from this metric. We have also obtained the conditions under which this change of metric leads a Berwald (or a Landsberg) space into a space of the same kind.

2 h-Randers conformal change

Let the Cartan's connection of Finsler space F^n be denoted by $CT = (F_{jk}^i, G_j^i, C_{jk}^i)$. Since $b_i(x, y)$ are components of h-vector, we have

$$(a) \quad b_i|_j = \dot{\partial}_j b_i - b_h C_{ij}^h = 0 \quad (b) \quad LC_{ij}^h b_h = \rho h_{ij} \quad (2.1)$$

Hence we obtain

$$\dot{\partial}_j b_i = L^{-1} \rho h_{ij} \quad (2.2)$$

Since h_{ij} are components of an indicatory tensor i.e. $h_{ij} y^j = 0$, we have $\dot{\partial}_i \beta = b_i$.

Definition 2.1. Let M^n be an n -dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function $L(x, y)$, ($y^i = \dot{x}^i$) of M^n . A change in the fundamental function L by the equation (1.1) on the same manifold M^n is called *h-Randers conformal change*. A space equipped with fundamental metric \bar{L} is called *h-Randers conformally changed Finsler space \bar{F}^n* .

Differentiating equation (1.1) with respect to y^i , the normalized supporting element $\bar{l}_i = \dot{\partial}_i \bar{L}$ is given by

$$\bar{l}_i = e^\sigma l_i + b_i, \quad (2.3)$$

where $l_i = \dot{\partial}_i L$ is the normalized supporting element l_i of F^n .

Differentiating (2.3) with respect to y^j and using (2.2) and the fact that $\dot{\partial}_j l_i = L^{-1} h_{ij}$, we get

$$\bar{h}_{ij} = \phi h_{ij}, \quad (2.4)$$

where $\phi = L^{-1} \bar{L}(e^\sigma + \rho)$ and $h_{ij} = L \dot{\partial}_i \dot{\partial}_j L$ is the angular metric tensor in the Finsler space F^n .

Since $h_{ij} = g_{ij} - l_i l_j$, from (2.3) and (2.4) the fundamental tensor $\bar{g}_{ij} = \dot{\partial}_i \dot{\partial}_j \frac{\bar{L}^2}{2} = \bar{h}_{ij} + \bar{l}_i \bar{l}_j$ is given as

$$\bar{g}_{ij} = \phi g_{ij} + b_i b_j + e^\sigma (b_i l_j + b_j l_i) + (e^{2\sigma} - \phi) l_i l_j \quad (2.5)$$

It is easy to see that the $\det(\bar{g}_{ij})$ does not vanish, and the reciprocal tensor with components \bar{g}^{ij} of \bar{F}^n , obtainable from $\bar{g}^{ij} \bar{g}_{jk} = \delta_k^i$, is given by

$$\bar{g}^{ij} = \phi^{-1} g^{ij} - \mu l^i l^j - \phi^{-2} (e^\sigma + \rho) (l^i b^j + l^j b^i), \quad (2.6)$$

where $\mu = (e^\sigma + \rho)^2 \phi^{-3} (e^\sigma - b^2 - \phi)$, $b^2 = b_i b^i$, $b^i = g^{ij} b_j$ and g^{ij} is the reciprocal tensor of g_{ij} of F^n .

We have following lemma [13]:

Lemma 2.1. *The scalar ρ used in the condition of h-vector is a function of coordinates x^i only.*

From equations (1.1), (2.3) and lemma 2.1 we have

$$\dot{\partial}_i \phi = L^{-1} (e^\sigma + \rho) m_i, \quad (2.7)$$

where

$$m_i = b_i - (L^{-1} \beta) l_i \quad (2.8)$$

Differentiating (2.4) with respect to y^k and using (2.3), (2.4), (2.7) and the relation $\dot{\partial}_k h_{ij} = 2C_{ijk} - L^{-1} (l_i h_{jk} + l_j h_{ik})$, the Cartan covariant tensor \bar{C}_{ijk} is given by

$$\bar{C}_{ijk} = \phi C_{ijk} + \frac{(e^\sigma + \rho)}{2L} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j), \quad (2.9)$$

where C_{ijk} is (h)hv-torsion tensor of Cartan's connection CT of Finsler space F^n .

From the definition of m_i , it is evident that

$$\begin{aligned} (a) \quad m_i l^i &= 0, & (b) \quad m_i b^i &= b^2 - \frac{\beta^2}{L^2} = b_i m^i, \\ (c) \quad g_{ij} m^i &= h_{ij} m^i = m_j, & (d) \quad C_{ihj} m^h &= L^{-1} \rho h_{ij} \end{aligned} \quad (2.10)$$

From (2.1), (2.6), (2.9) and (2.10), we get

$$\begin{aligned} \bar{C}_{ij}^h &= C_{ij}^h + \frac{1}{2L} (h_{ij} m^h + h_j^h m_i + h_i^h m_j) - \frac{1}{L} \left[\left\{ \rho + \frac{L}{2L} (b^2 - \frac{\beta^2}{L^2}) \right\} h_{ij} + \frac{L}{L} m_i m_j \right] l^h \end{aligned} \quad (2.11)$$

Proposition 2.1. *Let $\bar{F}^n = (M^n, \bar{L})$ be an n -dimensional Finsler space obtained from the h -Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the normalized supporting element \bar{l}_i , angular metric tensor \bar{h}_{ij} , fundamental metric tensor \bar{g}_{ij} and $(h)hv$ -torsion tensor \bar{C}_{ijk} of \bar{F}^n are given by (2.3), (2.4), (2.5) and (2.9) respectively.*

3 Geodesic Spray coefficients of \bar{F}^n

Let s be the arc-length of a curve $x^i = x^i(t)$ on a differentiable manifold M^n , then the equation of a geodesic [5] of $F^n = (M^n, L)$ is written in the well-known form:

$$\frac{d^2 x^i}{ds^2} + 2G^i(x, \frac{dx}{ds}) = 0, \quad (3.1)$$

where functions $G^i(x, y)$ are the geodesic spray coefficients given by

$$2G^i = g^{ir}(y^j \dot{\partial}_r \partial_j F - \partial_r F), \quad F = \frac{L^2}{2}.$$

Now suppose \bar{s} is the arc-length of a curve $\bar{x}^i = \bar{x}^i(t)$ on a differentiable manifold M^n in the Finsler space $\bar{F}^n = (M^n, \bar{L})$, then the equation of geodesic in \bar{F}^n can be written as

$$\frac{d^2 \bar{x}^i}{d\bar{s}^2} + 2\bar{G}^i(x, \frac{d\bar{x}}{d\bar{s}}) = 0, \quad (3.2)$$

where functions $\bar{G}^i(x, y)$ are given by

$$2\bar{G}^i = \bar{g}^{ir}(y^j \dot{\partial}_r \partial_j \bar{F} - \partial_r \bar{F}), \quad \bar{F} = \frac{\bar{L}^2}{2}.$$

Since $d\bar{s} = \bar{L}(x, dx)$, this is also written as

$$d\bar{s} = e^{\sigma(x)} L(x, dx) + b_i(x, y) dx^i = e^{\sigma(x)} ds + b_i(x, y) dx^i$$

Since $ds = L(x, dx)$, we have

$$\frac{dx^i}{ds} = \frac{dx^i}{d\bar{s}} [e^{\sigma(x)} + b_i(x, y) \frac{dx^i}{ds}] \quad (3.3)$$

Differentiating (3.3) with respect to s , we have

$$\frac{d^2 x^i}{ds^2} = \frac{d^2 \bar{x}^i}{d\bar{s}^2} [e^{\sigma(x)} + b_i \frac{dx^i}{ds}]^2 + \frac{dx^i}{d\bar{s}} \left(\frac{de^{\sigma(x)}}{ds} + \frac{db_i}{ds} \frac{dx^i}{ds} + b_i \frac{d^2 x^i}{ds^2} \right)$$

Substituting the value of $\frac{dx^i}{ds}$ from (3.3), the above equation becomes

$$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{ds^2} [e^{\sigma(x)} + b_i \frac{dx^i}{ds}]^2 + \frac{\frac{dx^i}{ds}}{[e^{\sigma(x)} + b_i \frac{dx^i}{ds}]} \left(\frac{de^{\sigma(x)}}{ds} + \frac{db_i}{ds} \frac{dx^i}{ds} + b_i \frac{d^2x^i}{ds^2} \right) \quad (3.4)$$

Now differentiating equation (1.1) with respect to x^i we have

$$\partial_i \bar{L} = e^{\sigma} A_i + B_i, \quad (3.5)$$

where $A_i = L \partial_i \sigma + \partial_i L$ and $B_i = \partial_i \{b_r(x, y)\} y^r$.

Differentiating above equation with respect to y^j we have

$$\dot{\partial}_j \partial_i \bar{L} = e^{\sigma} \dot{\partial}_j A_i + \dot{\partial}_j B_i, \quad (3.6)$$

where $\dot{\partial}_j A_i = l_j \partial_i \sigma + \dot{\partial}_j \partial_i L$ and $\dot{\partial}_j B_i = \dot{\partial}_j \{\partial_i b_r(x, y)\} y^r + \partial_i b_r(x, y) \delta_j^r$.

Since

$$2\bar{G}_r = y^j (\bar{l}_r \partial_j \bar{L} + \bar{L} \dot{\partial}_r \partial_j \bar{L}) - \bar{L} \partial_r \bar{L}$$

therefore using equations (2.3), (3.5) and (3.6) we have

$$\begin{aligned} 2\bar{G}_r = 2e^{2\sigma} G_r + y^j \{ 2e^{2\sigma} l_r L \dot{\partial}_j \sigma + e^{\sigma} (l_r B_j + b_r A_j) + b_r B_j + \\ e^{\sigma} L \dot{\partial}_r B_j + \beta e^{\sigma} \dot{\partial}_r A_j + \beta \dot{\partial}_r B_j \} - (e^{2\sigma} L^2 \partial_r \sigma + e^{\sigma} L B_r \\ + \beta e^{\sigma} A_r + \beta B_r), \end{aligned} \quad (3.7)$$

where $G_r = y^j \{ l_r \partial_j L + L \dot{\partial}_r \partial_j L \} - L \partial_r L$ is the spray coefficients for the Finsler space F^n .

Using equations (2.6) and (3.7) we have

$$\bar{G}^i = J G^i + M^i, \quad (3.8)$$

where $G^i = g^{ir} G_r$, $J = \frac{1}{\phi}$, and $M^i = \frac{1}{2} e^{2\sigma} G_r \{ -\mu l^i l^r - \phi^{-2} (e^{\sigma} + \rho) (l^i b^r + l^r b^i) \} + \frac{1}{2} [\phi^{-1} g^{ir} - \mu l^i l^r - \phi^{-2} (e^{\sigma} + \rho) (l^i b^r + l^r b^i)] [y^j \{ 2e^{2\sigma} l_r L \dot{\partial}_j \sigma + e^{\sigma} (l_r B_j + b_r A_j) + b_r B_j + e^{\sigma} L \dot{\partial}_r B_j + \beta e^{\sigma} \dot{\partial}_r A_j + \beta \dot{\partial}_r B_j \} - (e^{2\sigma} L^2 \partial_r \sigma + e^{\sigma} L B_r + \beta e^{\sigma} A_r + \beta B_r)]$.

Theorem 3.1. *Let $\bar{F}^n = (M^n, \bar{L})$ be an n -dimensional Finsler space obtained from the h -Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the geodesic spray coefficients \bar{G}^i for the Finsler space \bar{F}^n are given by (3.8) in the terms of the geodesic spray coefficients G^i of the Finsler space F^n .*

Corollary 3.1. *Let $\bar{F}^n = (M^n, \bar{L})$ be an n -dimensional Finsler space obtained from the h -Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the equation of geodesic of \bar{F}^n is given by (3.2), where $\frac{d^2 x^i}{ds^2}$ and \bar{G}^i are given by (3.4) and (3.8) respectively.*

4 C-reducibility of \bar{F}^n

Following Matsumoto [8], in this section we shall investigate special cases of the Finsler space with h -Randers conformally changed Finsler space \bar{F}^n .

Definition 4.1. *A Finsler space (M^n, L) with dimension $n \geq 3$ is said to be quasi-C-reducible if the Cartan tensor C_{ijk} satisfies*

$$C_{ijk} = Q_{ij}C_k + Q_{jk}C_i + Q_{ki}C_j, \quad (4.1)$$

where Q_{ij} is a symmetric indicatory tensor.

Substituting $h = j$ in equation (2.11) we get

$$\bar{C}_i = C_i + \frac{(n+1)}{2\bar{L}}m_i \quad (4.2)$$

Using equations (2.9) and (4.2), we have

$$\bar{C}_{ijk} = \phi C_{ijk} + \frac{\phi}{(n+1)}\pi_{(ijk)}\{h_{ij}(\bar{C}_k - C_k)\},$$

where $\pi_{(ijk)}$ represents cyclic permutation and sum over the indices i, j and k .

The above equation can be written as

$$\bar{C}_{ijk} = \phi C_{ijk} + \frac{\phi}{(n+1)}\pi_{(ijk)}(h_{ij}\bar{C}_k) - \frac{\phi}{(n+1)}\pi_{(ijk)}(h_{ij}C_k)$$

Thus

Lemma 4.1. *In an h -Randers conformally changed Finsler space \bar{F}^n , the Cartan's tensor can be written in the form*

$$\bar{C}_{ijk} = \pi_{(ijk)}(\bar{H}_{ij}\bar{C}_k) + V_{ijk}, \quad (4.3)$$

where $\bar{H}_{ij} = \frac{\bar{h}_{ij}}{(n+1)}$ and $V_{ijk} = \phi C_{ijk} - \frac{\phi}{(n+1)}\pi_{(ijk)}(h_{ij}C_k)$.

Since \bar{H}_{ij} is a symmetric and indicatory tensor, so from the above lemma and (4.1) we get

Theorem 4.1. *An h -Randers conformally changed Finsler space \bar{F}^n is quasi- C -reducible if the tensor V_{ijk} of equation (4.3) vanishes identically.*

We obtain a generalized form of Matsumoto's result known [8] as a corollary of the above theorem

Corollary 4.1. *If F^n is Riemannian then an h -Randers conformally changed Finsler space \bar{F}^n is always a quasi- C -reducible Finsler space.*

Definition 4.2. *A Finsler space (M^n, L) of dimension $n \geq 3$ is called C -reducible if the Cartan tensor C_{ijk} is written in the form*

$$C_{ijk} = \frac{1}{(n+1)}(h_{ij}C_k + h_{ki}C_j + h_{jk}C_i) \quad (4.4)$$

Now from equation (2.9) and definition of C -reducibility we have

$$\phi C_{ijk} = \pi_{(ijk)}(\bar{h}_{ij}N_k), \quad (4.5)$$

where $N_k = \frac{1}{(n+1)}\bar{C}_k - \frac{1}{2L}m_k$. Conversely, if (4.5) is satisfied for certain covariant vector N_k then from (2.9) we have

$$\bar{C}_{ijk} = \frac{1}{(n+1)}\pi_{(ijk)}(\bar{h}_{ij}\bar{C}_k) \quad (4.6)$$

Thus we have

Theorem 4.2. *An h -Randers conformally changed Finsler space \bar{F}^n is C -reducible iff equation (4.5) holds good.*

Corollary 4.2. *If the Finsler space F^n is C -reducible Finsler space, then an h -Randers conformally changed Finsler space \bar{F}^n is always a C -reducible Finsler space.*

5 Some Important tensors of \bar{F}^n

The v -curvature tensor [8] of Finsler space with fundamental function L is given by

$$S_{hijk} = C_{ijr}C_{hk}^r - C_{ikr}C_{hj}^r$$

Therefore the v -curvature tensor of an h -Randers conformally changed Finsler space \bar{F}^n will be given by

$$\bar{S}_{hijk} = \bar{C}_{ijr}\bar{C}_{hk}^r - \bar{C}_{ikr}\bar{C}_{hj}^r \quad (5.1)$$

From equations (2.9) and (2.11) we have

$$\begin{aligned} \bar{C}_{ijr}\bar{C}_{hk}^r = \phi[& C_{ijr}C_{hk}^r + (\frac{\rho}{L\bar{L}} - \frac{m^2}{4\bar{L}^2})h_{hk}h_{ij} + \frac{1}{2\bar{L}}(C_{ijk}m_h + \\ & C_{ijh}m_k + C_{ihk}m_j + C_{hjk}m_i) + \frac{1}{4\bar{L}^2}(h_{hj}m_im_k + \\ & h_{hi}m_jm_k + h_{jk}m_im_h + h_{ik}m_hm_j)], \end{aligned} \quad (5.2)$$

where $h_{jr}C_{hk}^r = C_{jhk} = h_j^r C_{rhk}$, $m_im^i = m^2$.

Using equations (5.1) and (5.2) we have

$$\begin{aligned} \bar{S}_{hijk} = \phi[& S_{hijk} + (\frac{\rho}{L\bar{L}} - \frac{m^2}{4\bar{L}^2})\{h_{hk}h_{ij} - h_{hj}h_{ik}\} + \frac{1}{4\bar{L}^2}\{h_{hj}m_im_k \\ & - h_{hk}m_im_j + h_{ik}m_hm_j - h_{ij}m_hm_k\}] \end{aligned} \quad (5.3)$$

Proposition 5.1. *In an h -Randers conformally changed Finsler space \bar{F}^n the v -curvature tensor \bar{S}_{hijk} is given by (5.3).*

It is well known[8] that the v -curvature tensor of any three-dimensional Finsler space is of the form

$$L^2 S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij}), \quad (5.4)$$

where scalar S in (5.4) is a function of x alone.

Owing to this fact M. Matsumoto defined the S3-like Finsler space as

Definition 5.1. *A Finsler space F^n ($n \geq 3$) is said to be S3-like Finsler space if the v -curvature tensor is of the form (5.4).*

The v -curvature tensor of any four-dimensional Finsler space may be written as [8]:

$$L^2 S_{hijk} = \Theta_{(jk)}\{h_{hj}K_{ki} + h_{ik}K_{hj}\}, \quad (5.5)$$

where K_{ij} is a $(0, 2)$ type symmetric Finsler tensor field which is such that $K_{ij}y^j = 0$ and the symbol $\Theta_{(jk)}\{...\}$ denotes the interchange of j, k and subtraction. The definition of S4-like Finsler space is given as

Definition 5.2. *A Finsler space F^n ($n \geq 4$) is said to be S4-like Finsler space if the v -curvature tensor is of the form (5.5).*

From equation (5.3) we have

Lemma 5.1. *The v-curvature tensor \bar{S}_{hijk} of a h-Randers conformally changed Finsler space can be written as*

$$\bar{S}_{hijk} = \bar{S}(\bar{h}_{hj}\bar{h}_{ik} - \bar{h}_{hk}\bar{h}_{ij}) + U_{hijk}, \quad (5.6)$$

where $\bar{S} = -\frac{1}{\phi}(\frac{\rho}{LL} - \frac{m^2}{4L^2})$ and $U_{hijk} = \phi[S_{hijk} + \frac{1}{4L^2}\{h_{hj}m_im_k - h_{hk}m_im_j + h_{ik}m_hm_j - h_{ij}m_hm_k\}]$

From lemma (5.1) and definition of S3-like Finsler space we have

Theorem 5.1. *An h-Randers conformally changed Finsler space \bar{F}^n is S3-like if the tensor U_{hijk} of equation (5.6) vanishes identically.*

From equation (5.3) we have

Lemma 5.2. *The v-curvature tensor \bar{S}_{hijk} of an h-Randers conformally changed Finsler space can also be written as*

$$\bar{S}_{hijk} = \Theta_{(jk)}(\bar{h}_{ij}K_{ij} + \bar{h}_{ik}K_{hj}) + \phi S_{hijk}, \quad (5.7)$$

where $K_{ij} = \frac{1}{4L^2}m_im_j - \frac{1}{2}(\frac{\rho}{LL} - \frac{m^2}{4L^2})h_{ij}$.

Thus from lemma(5.2) and definition of S4-like Finsler space we have

Theorem 5.2. *If the v-curvature tensor of Finsler space F^n vanishes identically then an h-Randers conformally changed Finsler space \bar{F}^n is S4-like Finsler space.*

Now we are concerned with $(v)hv$ -torsion tensor P_{ijk} . With respect to the Cartan's connection CT , $L_{|i} = 0$, $l_{i|j} = 0$, $h_{ij|k} = 0$ hold good [8].

Taking h-covariant derivative of the equation (2.9) we have

$$\begin{aligned} \bar{C}_{ijk|h} = L^{-1}\bar{L}(e^\sigma\sigma_{|h} + \rho_{|h})\{C_{ijk} + \frac{1}{2L}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)\} \\ + \phi\{C_{ijk|h} + \frac{1}{2L}(h_{ij}m_{k|h} + h_{jk}m_{i|h} + h_{ki}m_{j|h})\}, \end{aligned} \quad (5.8)$$

where $m_{i|h} = b_{i|h} - L^{-1}l_i b_{r|h} y^r$.

Lemma 5.3. *The h-covariant derivative of the Cartan tensor \bar{C}_{ijk} of an h-Randers conformally changed Finsler space \bar{F}^n can be written as*

$$\bar{C}_{ijk|h} = \phi C_{ijk|h} + V_{ijkh}, \quad (5.9)$$

where $V_{ijkh} = L^{-1}\bar{L}(e^\sigma\sigma_{|h} + \rho_{|h})\{C_{ijk} + \frac{1}{2L}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)\} + \frac{\phi}{2L}(h_{ij}m_{k|h} + h_{jk}m_{i|h} + h_{ki}m_{j|h})$.

The $(v)hv$ -torsion tensor P_{ijk} of the Cartan connection CT is written in the form

$$P_{ijk} = C_{ijk|0},$$

where the subscript '0' means the contraction with respect to the supporting element y^i .

From the equation (5.8), the $(v)hv$ -torsion tensor \bar{P}_{ijk} is given by

$$\begin{aligned} \bar{P}_{ijk} = \phi P_{ijk} + L^{-1} \bar{L} (e^\sigma \sigma_{|0} + \rho_{|0}) \{ C_{ijk} + \frac{1}{2\bar{L}} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) \} \\ + \frac{\phi}{2\bar{L}} \{ h_{ij} m_{k|0} + h_{jk} m_{i|0} + h_{ki} m_{j|0} \} \end{aligned} \quad (5.10)$$

Thus we have

Proposition 5.2. *The $(v)hv$ -torsion tensor \bar{P}_{ijk} of an h -Randers conformally changed Finsler space can be written in the form of (5.10).*

From the equation (5.10) we have

Lemma 5.4. *The $(v)hv$ -torsion tensor \bar{P}_{ijk} of an h -Randers conformally changed Finsler space can also be written as*

$$\bar{P}_{ijk} = \phi P_{ijk} + W_{ijk}, \quad (5.11)$$

where $W_{ijk} = L^{-1} \bar{L} (e^\sigma \sigma_{|0} + \rho_{|0}) \{ C_{ijk} + \frac{1}{2\bar{L}} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) \} + \frac{\phi}{2\bar{L}} \{ h_{ij} m_{k|0} + h_{jk} m_{i|0} + h_{ki} m_{j|0} \}$.

We have

Definition 5.3. *A Finsler space is called a Berwald space if $C_{ijk|h} = 0$ holds good.*

Definition 5.4. *A Finsler space is called a Landsberg space if $P_{ijk} = 0$ holds good.*

In view of above definition (5.3) and the lemma (5.3) we have

Theorem 5.3. *If a Finsler space F^n is a Berwald space and the tensor V_{ijkh} of equation (5.9) vanishes identically then an h -Randers conformally changed Finsler space \bar{F}^n is a Berwald space.*

In view of above definition (5.4) and the lemma (5.4) we have

Theorem 5.4. *If a Finsler space F^n is a Landsberg space and the tensor W_{ijk} of equation (5.11) vanishes identically then an h -Randers conformally changed Finsler space \bar{F}^n is a Landsberg space.*

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